

THE MURNAGHAN–NAKAYAMA RULE FOR k -SCHUR FUNCTIONS

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ABSTRACT. We prove the Murnaghan–Nakayama rule for k -Schur functions of Lapointe and Morse, that is, we give an explicit formula for the expansion of the product of a power sum symmetric function and a k -Schur function in terms of k -Schur functions. This is proved using the noncommutative k -Schur functions in terms of the nilCoxeter algebra introduced by Lam and the affine analogue of noncommutative symmetric functions of Fomin and Greene.

1. INTRODUCTION

The Murnaghan–Nakayama rule [10, 13, 14] is a combinatorial formula for the characters $\chi_\lambda(\mu)$ of the symmetric group in terms of ribbon tableaux. Under the Frobenius characteristic map, there exists an analogous statement on the level of symmetric functions, which follows directly from the formula

$$(1.1) \quad p_r s_\lambda = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)} s_\mu.$$

Here p_r is the r -th power sum symmetric function, s_λ is the Schur function labeled by partition λ , and the sum is over all partitions $\lambda \subseteq \mu$ for which μ/λ is a border strip of size r . Recall that a border strip is a connected skew shape without any 2×2 squares. The height $\text{ht}(\mu/\lambda)$ of a border strip μ/λ is one less than the number of rows.

In [3], Fomin and Greene develop the theory of Schur functions in noncommuting variables. In particular, they derive a noncommutative version of the Murnaghan–Nakayama rule [3, Theorem 1.3] for the nilCoxeter algebra (or more generally the local plactic algebra)

$$(1.2) \quad \mathbf{p}_r \mathbf{s}_\lambda = \sum_w (-1)^{\text{asc}(w)} w \mathbf{s}_\lambda,$$

where w is a hook word of length r . Here \mathbf{p}_r and \mathbf{s}_λ are the noncommutative analogues of the power sum symmetric function and the Schur function (introduced in Section 2). Consider w as a word in a totally ordered alphabet that consists of the indices of the generators of the algebra. The word w is a hook word if $w = b_l b_{l-1} \dots b_1 a_1 a_2 \dots a_m$ where

$$(1.3) \quad b_l > b_{l-1} > \dots > b_1 > a_1 \leq a_2 \leq \dots \leq a_m$$

and $\text{asc}(w) = m - 1$ is the number of ascents in w . Actually, by [3, Theorem 5.1] it can further be assumed that the support of w is an interval.

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In this paper, we derive a (noncommutative) Murnaghan–Nakayama rule for the k -Schur functions of Lapointe and Morse [9]. k -Schur functions form a basis for the ring $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k]$ spanned by the first k complete homogeneous symmetric functions h_r , which is a subring of the ring of symmetric functions Λ . Lapointe and Morse [9] gave a formula for a homogeneous symmetric function h_r times a k -Schur function (at $t = 1$) as

$$(1.4) \quad h_r s_\lambda^{(k)} = \sum_{\mu \in \mathcal{P}^{(k)}} s_\mu^{(k)},$$

where the sum is over all k -bounded partitions $\mu \in \mathcal{P}^{(k)}$ such that μ/λ is a horizontal r -strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r -strip. Here $\lambda^{(k)}$ denotes the k -conjugate of λ . Equation (1.4) is a simple analogue of the Pieri rule for usual Schur functions, called the k -Pieri rule. This formula can in fact be taken as the definition of k -Schur functions from which many of their properties can be derived. Conjecturally, the k -Pieri definition of the k -Schur functions is equivalent to the original definition by Lapointe, Lascoux, and Morse [5] in terms of atoms.

Lam [4] defined a noncommutative version of the k -Schur functions in the affine nilCoxeter algebra as the dual of the affine Stanley symmetric functions

$$(1.5) \quad F_w(X) = \sum_{a=(a_1, \dots, a_t)} \langle \mathbf{h}_{a_t}(u) \mathbf{h}_{a_{t-1}}(u) \cdots \mathbf{h}_{a_1}(u) \cdot 1, w \rangle x_1^{a_1} \cdots x_t^{a_t},$$

where the sum is over all compositions of $\text{len}(w)$ satisfying $a_i \in [0, k]$. Here

$$\mathbf{h}_r(u) = \sum_A u_A^{\text{dec}}$$

are the analogues of homogeneous symmetric functions in noncommutative variables, where the sum is over all r -subsets A of $[0, k]$ and u_A^{dec} is the product of the generators of the affine nilCoxeter algebra in cyclically decreasing order with indices appearing in A . We denote the noncommutative analogue of $\Lambda_{(k)}$ by $\mathbf{\Lambda}_{(k)}$ as the subalgebra of the affine nilCoxeter algebra generated by these analogues of homogeneous symmetric functions. See Section 2.3 for further details.

Denote by $\mathbf{s}_\lambda^{(k)}$ the noncommutative k -Schur function labeled by the k -bounded partition λ and \mathbf{p}_r the noncommutative power sum symmetric function in the affine nilCoxeter algebra. There is a natural bijection from k -bounded partitions λ to $(k+1)$ -cores, denoted $\text{core}_{k+1}(\lambda)$ (see Section 2.1). We define a vertical domino in a skew-partition to be a pair of cells in the diagram, with one sitting directly above the other. For the skew of two k -bounded partitions $\lambda \subseteq \mu$ we define the height as

$$(1.6) \quad \text{ht}(\mu/\lambda) = \text{number of vertical dominos in } \mu/\lambda.$$

For ribbons, that is skew shapes without any 2×2 squares, the definition of height can be restated as the number of occupied rows minus the number of connected components. Notice that this is compatible with the usual definition of the height of a border strip.

Definition 1.1. The skew of two k -bounded partitions, μ/λ , is called a k -ribbon of size r if μ and λ satisfy the following properties:

- (0) (containment condition) $\lambda \subseteq \mu$ and $\lambda^{(k)} \subseteq \mu^{(k)}$;

- (1) (size condition) $|\mu/\lambda| = r$;
- (2) (ribbon condition) $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ is a ribbon;
- (3) (connectedness condition) $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ is k -connected (see Definition 2.3);
- (4) (height statistics condition) $\text{ht}(\mu/\lambda) + \text{ht}(\mu^{(k)}/\lambda^{(k)}) = r - 1$.

Our main result is the following theorem.

Theorem 1.2. *For $1 \leq r \leq k$ and λ a k -bounded partition, we have*

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)} \mathbf{s}_\mu^{(k)},$$

where the sum is over all k -bounded partitions μ such that μ/λ is a k -ribbon of size r .

If k is sufficiently large, then μ/λ satisfies Definition 1.1 is equivalent to μ/λ is a connected ribbon of size r (as a skew partition). Hence for k sufficiently large, Theorem 1.2 implies Equation (1.1).

Let λ, ν be k -bounded partitions of the same size and ℓ the length of ν . A k -ribbon tableau of shape λ and type ν is a filling, T , of the cells of λ with the labels $\{1, 2, \dots, \ell\}$ which satisfies the following conditions for all $1 \leq i \leq \ell$:

- (i) the shape of the restriction of T to the cells labeled $1, \dots, i$ is a partition, and
- (ii) the skew shape r_i , which is the restriction of T to the cells labeled i , is a k -ribbon of size ν_i .

We also define

$$\chi_{\lambda, \nu}^{(k)} = \sum_T \left(\prod_{i=1}^{\ell} (-1)^{\text{ht}(r_i)} \right),$$

where the sum is over all k -ribbon tableaux T of shape λ and type ν .

Iterating Theorem 1.2 gives the following corollary. We remark that this formula may also be considered as a definition of the k -Schur functions.

Corollary 1.3. *For ν a k -bounded partition, we have*

$$\mathbf{p}_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \chi_{\lambda, \nu}^{(k)} \mathbf{s}_\lambda^{(k)}.$$

All notation and definitions regarding our main Theorem 1.2 are given in Section 2. In Section 2 we also see that there is a ring isomorphism

$$\iota : \Lambda_{(k)} \rightarrow \Lambda_{(k)}$$

sending the noncommutative symmetric functions to their symmetric function counterpart. This leads us to the following corollary.

Corollary 1.4. *Theorem 1.2 and Corollary 1.3 also hold when replacing \mathbf{p}_r by the power sum symmetric function p_r , and $\mathbf{s}_\lambda^{(k)}$ by the k -Schur function $s_\lambda^{(k)}$.*

Dual k -Schur functions $\mathfrak{S}_\lambda^{(k)}$ indexed by k -bounded partitions λ form a basis of the quotient space $\Lambda^{(k)} = \Lambda / \langle p_r \mid r > k \rangle = \Lambda / \langle m_\lambda \mid \lambda_1 > k \rangle$ (they correspond to the affine Stanley symmetric functions indexed by Grassmannian elements). The Hall inner product $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Q}$ defined by $\langle h_\lambda, m_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$, can be restricted to $\langle \cdot, \cdot \rangle : \Lambda_{(k)} \times \Lambda^{(k)} \rightarrow \mathbb{Q}$, so that $s_\lambda^{(k)}$ and $\mathfrak{S}_\mu^{(k)}$ form dual bases $\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)} \rangle = \delta_{\lambda, \mu}$. Let z_λ be the size of the centralizer of any permutation of cycle type λ . Then $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda, \mu}$.

Corollary 1.5. *For ν a k -bounded partition, we have*

$$\mathfrak{S}_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \chi_{\nu, \lambda}^{(k)} p_\lambda.$$

Proof. Denote by $b_{\nu, \lambda}^{(k)}$ the coefficient of p_λ in $\mathfrak{S}_\nu^{(k)}$, that is, $\mathfrak{S}_\nu^{(k)} = \sum_\lambda b_{\nu, \lambda}^{(k)} p_\lambda$. Then, using Corollary 1.3 we have

$$z_\lambda b_{\nu, \lambda}^{(k)} = \langle p_\lambda, \mathfrak{S}_\nu^{(k)} \rangle = \left\langle \sum_\mu \chi_{\mu, \lambda}^{(k)} s_\mu^{(k)}, \mathfrak{S}_\nu^{(k)} \right\rangle = \chi_{\nu, \lambda}^{(k)}. \quad \square$$

Since the product of two k -bounded power symmetric functions is again a k -bounded power symmetric function, the expansion of the dual k -Schur functions in terms of p_λ of Corollary 1.5 is better suited for multiplication than the expansion in terms of monomial symmetric functions. The product of two k -bounded monomial symmetric functions is a sum of monomial symmetric functions which are not necessarily k -bounded.

The classical Murnaghan-Nakayama rule (corresponding to sufficiently large k) has implications for representation theory. The well-known Frobenius map sends a representation V of the symmetric group S_n to the symmetric function

$$\sum_\mu \frac{\chi_V(\mu)}{z_\mu} p_\mu,$$

where $\chi_V(\mu)$ is the character of V evaluated on the conjugacy class of type μ . This map sends the irreducible representation V_λ to the Schur function s_λ . Therefore, whenever a Schur-positive symmetric function is expanded in terms of the power-sum basis, the coefficients can be interpreted as the character of some corresponding representation. However, this does not apply to Corollary 1.5, since the functions $\mathfrak{S}_\lambda^{(k)}$ are not Schur positive. The possibility of a different form of a k -Murnaghan-Nakayama rule that would have representation theoretical implications is discussed in Section 5.

The paper is organized as follows. In Section 2 we introduce all notation and definitions. In particular, we define the various noncommutative symmetric functions. In Section 3 we prove Theorem 3.1, which is the analogue of Theorem 1.2 formulated in terms of the nilCoxeter algebra. In Section 4 it is shown that Theorems 1.2 and 3.1 are equivalent. We conclude in Section 5 with some related open questions. In Appendices A and B we list some tables for $\chi_{\lambda, \mu}^{(k)}$ and its dual version $\tilde{\chi}_{\lambda, \mu}^{(k)}$.

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2. NOTATION

In this section we give all necessary definitions.

2.1. Partitions and cores. A sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a partition if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$. We say that ℓ is the length of λ and $|\lambda| = \lambda_1 + \dots + \lambda_\ell$ is its size. A partition λ is k -bounded if $\lambda_1 \leq k$. We denote by $\mathcal{P}^{(k)}$ the set of all k -bounded partitions.

One may represent a partition λ by its partition diagram, which contains λ_i boxes in row i . The conjugate λ^t corresponds to the diagram with rows and columns interchanged. We use French convention and label rows in decreasing order from bottom to top. For example



correspond to the partition $(3, 1)$ and its conjugate $(2, 1, 1)$, respectively.

For two partitions λ and μ whose diagrams are contained, that is $\lambda \subseteq \mu$, we denote by μ/λ the skew partition consisting of the boxes in μ not contained in λ . A ribbon is a skew shape which does not contain any 2×2 squares. An r -border strip is a connected ribbon with r boxes.

A partition λ is an r -core if no r -border strip can be removed from λ such that the result is again a partition. For example



is a 4-core. We denote the set of all r -cores by \mathcal{C}_r .

For a cell $c = (i, j) \in \lambda$ in row i and column j we define its hook length to be the number of cells in row i of λ to the right of c plus the number of cells in column j of λ weakly above c (including c). An alternative definition of an r -core is a partition without any cells of hook length equal to a multiple of r [12, Ch. 1, Ex. 8]. The content of cell $c = (i, j)$ is given by $j - i \pmod{r}$.

There exists a bijection [8]

$$(2.2) \quad \text{core}_{k+1} : \mathcal{P}^{(k)} \rightarrow \mathcal{C}_{k+1}$$

from k -bounded partitions to $(k+1)$ -cores defined as follows. Let $\lambda \in \mathcal{P}^{(k)}$ considered as a set of cells. Starting from the smallest row, check whether there are any cells of hook length greater than k . If so, slide the row and all those in the rows below to the right by

the minimal amount so that none of cells in that row have a hook length greater than k . Then continue the procedure with the rows below. The positions of the cells define a skew partition and the outer partition is a $(k+1)$ -core.

The inverse map $\text{core}_{k+1}^{-1} : \mathcal{C}_{k+1} \rightarrow \mathcal{P}^{(k)}$ is slightly easier to compute. The partition $\text{core}_{k+1}^{-1}(\kappa)$ is of the same length as the $(k+1)$ -core κ and the i^{th} entry of the partition is the number of cells in the i^{th} row of κ which have a hook smaller or equal to k .

Let $\lambda \in \mathcal{P}^{(k)}$. Then the k -conjugate $\lambda^{(k)}$ of λ is defined as $\text{core}_{k+1}^{-1}(\text{core}_{k+1}(\lambda)^t)$.

Example 2.1. For $k = 3$, take $\lambda = (3, 2, 1, 1) \in \mathcal{P}^{(k)}$ so that

$$\text{core}_4 : \begin{array}{|c|c|c|} \hline \square \\ \square \\ \square \\ \square \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

which is the 4-core in (2.1) (where we have drawn the original boxes of λ in bold). To obtain the k -conjugate $\lambda^{(3)}$ of λ we calculate

$$\text{core}_4^{-1} : \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}.$$

2.2. Affine nilCoxeter algebra. The affine nilCoxeter algebra \mathcal{A}_k is the algebra over \mathbb{Z} generated by u_0, u_1, \dots, u_k satisfying

$$(2.3) \quad \begin{aligned} u_i^2 &= 0 & \text{for } i \in [0, k], \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} & \text{for } i \in [0, k], \\ u_i u_j &= u_j u_i & \text{for } i, j \in [0, k] \text{ such that } |i - j| \geq 2, \end{aligned}$$

where all indices are taken modulo $k+1$. We view the indices $i \in [0, k]$ as living on a circle, with node i being adjacent to nodes $i-1$ and $i+1$ (modulo $k+1$). As with Coxeter groups, we have a notion of reduced words of elements $u \in \mathcal{A}_k$ as the shortest expressions in the generators. If $u = u_{i_1} \cdots u_{i_m}$ is a reduced expression, we call $\{i_1, \dots, i_m\}$ the support of u denoted $\text{supp}(u)$ (which is independent of the reduced word and only depends on u itself). Also, $i_1 \dots i_m$ is the corresponding reduced word and $\text{len}(u) = m$ is the length of u .

A word w in the letters $[0, k]$ is cyclically decreasing (resp. increasing) if the length of w is at most k , every letter appears at most once, and if $i, i-1 \in w$ then i occurs before (resp. after) $i-1$. Note that since u_i and u_j commute if i is not adjacent to j , all cyclically decreasing (resp. increasing) words w with the same support give rise to the same affine nilCoxeter group element $\prod_{i \in w} u_i$. For a proper subset $A \subsetneq [0, k]$ we define $u_A^{\text{dec}} \in \mathcal{A}_k$ (resp. $u_A^{\text{inc}} \in \mathcal{A}_k$) to be the element corresponding to cyclically decreasing (resp. increasing) words with support A .

Example 2.2. Take $k = 6$ and $A = \{0, 2, 3, 4, 6\}$. Then $u_A^{\text{dec}} = (u_0 u_6)(u_4 u_3 u_2) = (u_4 u_3 u_2)(u_0 u_6)$ and $u_A^{\text{inc}} = (u_6 u_0)(u_2 u_3 u_4) = (u_2 u_3 u_4)(u_6 u_0)$.

If $u \in \mathcal{A}_k$ is supported on a proper subset S of $[0, k]$, then we specify a canonical interval I_S which contains the subset S . Identify the smallest element a (from the numbers 0 through k with the integer order) which does not appear in S . Then the canonical cyclic interval which we choose orders the elements

$$a + 1 < a + 2 < \cdots < k < 0 < 1 < \cdots < a - 1,$$

(where we identify k and -1 when necessary).

Definition 2.3. An element $u \in \mathcal{A}_k$ (resp. word w) is k -connected if its support S is an interval in I_S .

Example 2.4. For $k = 6$, the word $w = 0605$ is k -connected, whereas $w = 06052$ is not.

Suppose $u \in \mathcal{A}_k$ has support $S \subsetneq [0, k]$. We say that u corresponds to a hook word if it has a reduced word w of the form of Equation (1.3) with respect to the canonical order I_S . In this case we denote by $\text{asc}(u)$ or $\text{asc}(w)$ the number of ascents $\text{asc}_{I_S}(w)$ in the canonical order.

Example 2.5. Take $u = u_3 u_2 u_6 u_0 u_4 \in \mathcal{A}_6$. In this case $S = \{0, 2, 3, 4, 6\}$ and I_S is given by $2 < 3 < 4 < 5 < 6 < 0$. The word $w = (3)(2460)$ is a hook word with respect to I_S and $\text{asc}(u) = 3$.

The generators u_i in the nilCoxeter algebra \mathcal{A}_k act on a $(k+1)$ -core $\nu \in \mathcal{C}_{k+1}$ by

$$(2.4) \quad u_i \cdot \nu = \begin{cases} \nu \text{ with all corner cells of content } i \text{ added if they exist,} \\ 0 \text{ otherwise.} \end{cases}$$

This action is extended to the rest of the algebra \mathcal{A}_k and can be shown to be consistent with the relations of the generators. Under the bijection core_{k+1}^{-1} to k -bounded partitions only the topmost box added to diagram survives. The action of u_i on a k -bounded partition λ under core_{k+1} is denoted $u_i \cdot \lambda$.

Example 2.6. Taking $\nu = \text{core}_4(\lambda)$ from Example 2.1 we obtain

$$u_2 \cdot \nu = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \quad \text{and} \quad \text{core}_4^{-1}(u_2 \cdot \nu) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

where the boxes added by u_2 of content 2 are indicated in bold.

2.3. Noncommutative symmetric functions. We now give the definition of the noncommutative symmetric functions \mathbf{e}_r , \mathbf{h}_r , $\mathbf{s}_{(r-i, 1^i)}$, \mathbf{p}_r , and $\mathbf{s}_\lambda^{(k)}$ in terms of the affine nilCoxeter algebra.

Following Lam [4], for $r = 1, \dots, k$, we define the noncommutative homogeneous symmetric functions

$$\mathbf{h}_r = \sum_{A \in \binom{[0, k]}{r}} u_A^{\text{dec}},$$

where u_A^{dec} is a cyclically decreasing element with support A as defined in Section 2.2. We take as a defining relation for the elements \mathbf{e}_r the equation $\sum_{i=0}^r (-1)^i \mathbf{e}_{r-i} \mathbf{h}_i = 0$. It can be shown [4, Proposition 16] that then

$$\mathbf{e}_r = \sum_{A \in \binom{[0,k]}{r}} u_A^{\text{inc}},$$

where u_A^{inc} is a cyclically increasing element with support A . More generally, the hook Schur functions for $r \leq k$ are given by

$$\mathbf{s}_{(r-i, 1^i)} = \mathbf{h}_{r-i} \mathbf{e}_i - \mathbf{h}_{r-i+1} \mathbf{e}_{i-1} + \cdots + (-1)^i \mathbf{h}_r$$

and we will demonstrate in Corollary 3.5 (below) that these elements may also be expressed as a sum over certain words.

The noncommutative power sum symmetric functions for $1 \leq r \leq k$ are defined through the analogue of a classical identity with ribbon Schur functions

$$\mathbf{p}_r = \sum_{i=0}^{r-1} (-1)^i \mathbf{s}_{(r-i, 1^i)}.$$

Lam [4, Proposition 8] proved that, even though the variables u_i do not commute, the elements \mathbf{h}_r for $1 \leq r \leq k$ commute and consequently, so do the other elements \mathbf{e}_r , \mathbf{p}_r , $\mathbf{s}_{(r-i, 1^i)}$ we have defined in terms of the \mathbf{h}_r . We define $\mathbf{\Lambda}_{(k)} = \mathbb{Z}[\mathbf{h}_1, \dots, \mathbf{h}_k]$ to be the noncommutative analogue of $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k]$.

We define the noncommutative k -Schur functions $\mathbf{s}_\lambda^{(k)}$ by the noncommutative analogue of the k -Pieri rule (1.4). Let us denote by $\mathcal{H}_r^{(k)}$ the set of all pairs (μ, λ) of k -bounded partitions μ, λ such that μ/λ is a horizontal r -strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r -strip (which describes the summation in the k -Pieri rule). Then for a k -bounded partition λ we require that

$$(2.5) \quad \mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu: (\mu, \lambda) \in \mathcal{H}_r^{(k)}} \mathbf{s}_\mu^{(k)}.$$

This definition can be used to expand the \mathbf{h}_μ elements in terms of the elements $\mathbf{s}_\lambda^{(k)}$. The transition matrix is described by the number of k -tableaux of given shape and weight (see [8]). Since this matrix is unitriangular, this system of relations can be inverted over the integers and hence $\{\mathbf{s}_\lambda^{(k)} \mid \lambda \in \mathcal{P}^{(k)}\}$ forms a basis of $\mathbf{\Lambda}_{(k)}$.

As shown in [8, 6], for $1 \leq r \leq k$, we have if $(\mu, \lambda) \in \mathcal{H}_r^{(k)}$, then there is a cyclically decreasing element $u \in \mathcal{A}_k$ of length r such that $\mu = u \cdot \lambda$. Moreover, if $u \in \mathcal{A}_k$ is cyclically decreasing and $\mu = u \cdot \lambda \neq 0$, then $(\mu, \lambda) \in \mathcal{H}_r^{(k)}$.

Example 2.7. Take $\lambda = (3, 3, 1, 1) \in \mathcal{P}^{(3)}$ and $u = u_0 u_3$. Then

$$\text{core}_4(\lambda) = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \quad \text{and} \quad u \cdot \text{core}_4(\lambda) = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

so that $((3, 3, 2, 1, 1), (3, 3, 1, 1)) \in \mathcal{H}_2^{(3)}$.

Hence, we may rewrite (2.5) as

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu: (\mu, \lambda) \in \mathcal{H}_r^{(k)}} \mathbf{s}_\mu^{(k)} = \sum_{A \in \binom{[0, k]}{r}} \mathbf{s}_{u_A^{\text{dec}} \cdot \lambda}^{(k)},$$

where we assume $\mathbf{s}_{u_A^{\text{dec}} \cdot \lambda}^{(k)} = 0$ if $u_A^{\text{dec}} \cdot \lambda = 0$. The elements $\mathbf{h}_r = \sum_{A \in \binom{[0, k]}{r}} u_A^{\text{dec}}$ generate $\Lambda_{(k)}$, and therefore more generally for any element $\mathbf{f} = \sum_u c_u u \in \Lambda_{(k)}$ with $u \in \mathcal{A}_k$ and $c_u \in \mathbb{Z}$

$$(2.6) \quad \mathbf{f} \mathbf{s}_\lambda^{(k)} = \sum_u c_u \mathbf{s}_{u \cdot \lambda}^{(k)}.$$

Since all of the noncommutative symmetric functions in this section commute and satisfy the same defining relations as their commutative counterparts, there is a ring isomorphism

$$\iota : \Lambda_{(k)} \rightarrow \Lambda_{(k)}$$

sending $\mathbf{h}_r \mapsto h_r$, $\mathbf{e}_r \mapsto e_r$, $\mathbf{p}_r \mapsto p_r$, $\mathbf{s}_\lambda^{(k)} \mapsto s_\lambda^{(k)}$.

3. MAIN RESULT: MURNAGHAN–NAKAYAMA RULE IN TERMS OF WORDS

We now restate Theorem 1.2 in terms of the action of words. This result is proved in the remainder of this section.

Theorem 3.1. *For $1 \leq r \leq k$ and λ a k -bounded partition, we have*

$$(3.1) \quad \mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_{(w, \mu)} (-1)^{\text{asc}(w)} \mathbf{s}_\mu^{(k)},$$

where the sum is over all pairs (w, μ) of reduced words w in the affine nilCoxeter algebra \mathcal{A}_k and k -bounded partitions μ satisfying

- (1') (size condition) $\text{len}(w) = r$;
- (2') (ribbon condition) w is a hook word;
- (3') (connectedness condition) w is k -connected;
- (4') (weak order condition) $\mu = w \cdot \lambda$.

In Section 4 we will show the equivalence of Theorem 1.2 and Theorem 3.1.

The proof of Theorem 3.1 essentially amounts to computing an expression for \mathbf{p}_r in terms of words. Since all words involved will be of length $\leq k$, there will be a canonical order on the support as introduced in Section 2.2. The statistic $\text{asc}(w)$, and the property of being a hook word, will always be in terms of this canonical ordering.

Lemma 3.2. For $0 \leq i \leq r \leq k$,

$$(3.2) \quad \mathbf{h}_{r-i} \mathbf{e}_i = \sum_w w ,$$

where the sum is over all words w satisfying (1'), (2') with respect to the canonical order, and $\text{asc}(w) \in \{i-1, i\}$.

Proof. \mathbf{h}_{r-i} is the sum over all cyclically decreasing nilCoxeter group elements of length $r-i$ and \mathbf{e}_i is the sum over all cyclically increasing nilCoxeter group elements of length i . Hence

$$\mathbf{h}_{r-i} \mathbf{e}_i = \sum_{\substack{(u,v) \\ u \text{ cycl. dec., } |u| = r-i \\ v \text{ cycl. inc., } |v| = i}} uv.$$

Rearrange each u and v so that they together form a hook with respect to the canonical order associated to the set $\text{supp}(u) \cup \text{supp}(v)$. Either the last letter of u is smaller than the first letter of v , in which case the total ascent is i , or the last letter of u is bigger than the first letter in v , in which case the total number of ascents is $i-1$. This yields a bijection between hook words in the canonical order and pairs appearing in this sum with the number of ascents in $\{i, i-1\}$. In the corner case $i=0$ (resp. $i=r$) the number of ascents can only be 0 (resp. $r-1$ due to the fact that the words are of length r). \square

Example 3.3. Take $k=8$, $u = (u_1 u_0 u_8)(u_5 u_4)$ and $v = (u_2 u_3)(u_0)$, so that $i=3$ and $r=8$. In this case the canonical order is $7 < 8 < 0 < 1 < 2 < 3 < 4 < 5$ and we would write uv as $uv = [(u_5 u_4)(u_1 u_0 u_8)][(u_0)(u_2 u_3)]$, giving rise to the word $w = (5410)(8023)$ with $i=3$ ascents. If on the other hand $u = (u_1 u_0)(u_5 u_4)$ and $v = (u_2 u_3)(u_8 u_0)$, so that $i=4$ and $r=8$, then we would write $uv = [(u_5 u_4)(u_1 u_0)][(u_8 u_0)(u_2 u_3)]$, giving rise to the word $w = (5410)(8023)$ with $i-1=3$ ascents.

Remark 3.4. Note that there may be multiplicities in (3.2) with respect to affine nilCoxeter group elements because there may be several hook words with the same number of ascents that are equivalent to the same affine nilCoxeter element. For example, $(4)(20)$ and $(0)(24)$ are two different hook words with exactly one ascent with respect to the interval $I_{\{0,2,4\}} = \{2 < 4 < 0\}$. Of course, they both correspond to the same affine nilCoxeter element since all letters in the word commute. The element with $u = u_2$ and $v = u_4 u_0$ would give rise to the hook word $w = (240)$ with 2 ascents.

We can use this lemma to get an expression for hook Schur functions.

Corollary 3.5. For $0 \leq i \leq r \leq k$, the hook Schur function is

$$\mathbf{s}_{(r-i, 1^i)} = \sum_w w ,$$

where the sum is over all words w satisfying (1'), (2') with respect to the canonical order, and $\text{asc}(w) = i$.

Proof. From our definition of the noncommutative Schur functions indexed by a hook partition, it follows that

$$\mathbf{s}_{(r-i, 1^i)} = \mathbf{h}_{r-i} \mathbf{e}_i - \mathbf{h}_{r-i+1} \mathbf{e}_{i-1} + \cdots + (-1)^i \mathbf{h}_r.$$

Hence by Lemma 3.2 the only words which do not appear in two terms with opposite signs are those that have $\text{asc}(w) = i$, which implies the corollary. \square

Example 3.6. Let $k = 3$. Then for $r = 3$ and $i = 1$ we have

$$\begin{aligned} \mathbf{s}_{2,1} = & u_1 u_0 u_1 + u_2 u_1 u_2 + u_3 u_2 u_3 + u_0 u_3 u_0 \\ & + u_1 u_3 u_0 + u_1 u_0 u_2 + u_2 u_0 u_1 + u_2 u_1 u_3 + u_3 u_1 u_2 + u_3 u_2 u_0 + u_0 u_2 u_3 + u_0 u_3 u_1. \end{aligned}$$

We can now write an expression for \mathbf{p}_r by using the definition.

Corollary 3.7. For $1 \leq r \leq k$,

$$\mathbf{p}_r = \sum_w (-1)^{\text{asc}(w)} w,$$

where the sum is over all words w satisfying (1') and (2') in the canonical order.

Proof. This follows immediately from the definition

$$\mathbf{p}_r = \sum_{i=0}^{r-1} (-1)^i \mathbf{s}_{(r-i, 1^i)}. \quad \square$$

In fact, we may restrict our attention to those words in the sum also satisfying (3') because it is possible to show that those not satisfying (3') will cancel.

Lemma 3.8. For $r \leq k$,

$$\mathbf{p}_r = \sum_w (-1)^{\text{asc}(w)} w,$$

where the sum is over all words w satisfying (1'), (2'), and (3').

Proof. Since each canonical interval can be viewed as an interval of the finite nilCoxeter group, the sign-reversing involution described before [3, Theorem 5.1] still holds and there is a sign-reversing involution on the terms which do not satisfy (3'). Hence it suffices to sum only over terms which are connected cyclic intervals. \square

Example 3.9. Let $k = 3$. Then

$$\mathbf{p}_2 = u_1 u_0 + u_2 u_1 + u_3 u_2 + u_0 u_3 - (u_1 u_2 + u_2 u_3 + u_3 u_0 + u_0 u_1).$$

Theorem 3.1 now follows from the action of words on $\mathbf{s}_\lambda^{(k)}$ given by Equation (2.6).

4. EQUIVALENCE OF MAIN THEOREMS

To show the equivalence of Theorems 1.2 and 3.1, we will show that a k -bounded partition μ satisfies conditions (0) through (4) of Definition 1.1 if and only if there exists a unique w such that the pair (w, μ) satisfies conditions (1') through (4') of Theorem 3.1, and that such a w will satisfy $\text{asc}(w) = \text{ht}(\mu/\lambda)$.

4.1. Primed implies unprimed. We begin by showing that conditions (1') through (4') of Theorem 3.1 imply conditions (0) through (4) of Definition 1.1.

The first two lemmas will be important to show the correspondence between ascents in hook words $\text{asc}(w)$ and the height of vertical strips $\text{ht}(\mu/\lambda)$, and also for the understanding of the statistics in condition (4) of Definition 1.1.

Lemma 4.1. *Let $u \in \mathcal{A}_k$ with $\text{supp}(u) \subsetneq [0, k]$ and let \mathcal{I} be the canonical interval with respect to $\text{supp}(u)$.*

- (1) *Suppose u has a reduced word $\gamma_1 \dots \gamma_m \delta_1 \dots \delta_\ell$ such that $\gamma_1 < \dots < \gamma_m > \delta_1 > \dots > \delta_\ell$ in \mathcal{I} . Then u also has a reduced word $\beta_1 \dots \beta_\ell \alpha_1 \dots \alpha_m$ such that $\beta_1 > \dots > \beta_\ell > \alpha_1 < \dots < \alpha_m$ in \mathcal{I} .*
- (2) *If u is k -connected and has a reduced word which is a hook word in \mathcal{I} , then this hook word is unique.*

Proof. The statement (1) follows directly from the Edelman-Greene insertion [2] by induction on ℓ . We think of the two reduced words as the following hook tableaux

$$\begin{array}{|c|c|c|} \hline \gamma_1 & \dots & \gamma_m \\ \hline & & \delta_1 \\ & & \vdots \\ & & \delta_\ell \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline \beta_1 \\ \hline \vdots \\ \hline \beta_\ell \\ \hline \alpha_1 & \dots & \alpha_m \\ \hline \end{array}.$$

For $\ell = 0$, the statement is trivial since $\gamma_1 \dots \gamma_m = \alpha_1 \dots \alpha_m$ satisfies both sets of required inequalities. Suppose that the statement is true for $\gamma_1 \dots \gamma_m \delta_1 \dots \delta_{\ell-1}$ for $\ell > 0$, namely that this word is equivalent to $\beta_1 \dots \beta_{\ell-1} \alpha'_1 \dots \alpha'_m$ with $\beta_1 > \dots > \beta_{\ell-1} > \alpha'_1 < \dots < \alpha'_m$. Now insert δ_ℓ in the following way: Let α'_i be smallest such that $\delta_\ell < \alpha'_i$. If $\alpha'_{i-1} \alpha'_i \delta_\ell = a(a+1)a$ for some $a \in \mathcal{I}$, then set $\beta_\ell = \alpha'_i$ and $\alpha_j = \alpha'_j$ for all $1 \leq j \leq m$. Otherwise set $\beta_\ell = \alpha'_i$, $\alpha_i = \delta_\ell$ and $\alpha_j = \alpha'_j$ for $j \neq i$. It is not hard to see that if $\gamma_1 \dots \gamma_m \delta_1 \dots \delta_{\ell-1}$ and $\beta_1 \dots \beta_{\ell-1} \alpha'_1 \dots \alpha'_m$ are equivalent, then $\gamma_1 \dots \gamma_m \delta_1 \dots \delta_\ell$ and $\beta_1 \dots \beta_\ell \alpha_1 \dots \alpha_m$ are also equivalent and all required inequalities are satisfied. (For the inequality $\beta_\ell < \beta_{\ell-1}$, observe that $\beta_{\ell-1} > \delta_{\ell-1} \geq \alpha'_i = \beta_\ell$.)

Statement (2) follows in a similar way as [11, Lemma 6.8] by induction on $\text{len}(u)$. For $\text{len}(u) \leq 3$, the uniqueness of the hook word follows directly from the braid relations (2.3). Now let $\text{len}(u) > 3$, w a hook word for u , and M the maximal letter in $\text{supp}(u)$. There are two cases: either w contains one or two letters M .

First assume that w contains one M . Then $w = Mv$ or vM . Without loss of generality we may assume that $w = Mv$ as the other case is similar. Then v is a k -connected hook word with $\text{len}(v) < \text{len}(w)$, so that by induction v is unique. By the form of the braid relations (2.3), every reduced word for u must have a single M which precedes all $M-1$ (since w does not contain any $M+1$ and the only way to obtain two M s is to use the braid relation $(M-1)M(M-1) \equiv M(M-1)M$, but there is no $M-1$ to the left of M). Hence $w = Mv$ is the unique hook word.

Now assume that w contains two M s, so that $w = MvM$. Suppose that w' is another hook word for u . Then w' must contain an M at the beginning or the end. Assume without

loss of generality that $w' = Mv'$. Again by induction, the hook word v' is unique. Since vM is also a hook word equivalent to v' , we must have that $v' = vM$, which implies that $w' = w$. \square

Lemma 4.2. *Let $\lambda \in \mathcal{P}^{(k)}$ and $u \in \mathcal{A}_k$ with reduced word $(a-1)wa$, where w contains neither a nor $a-1$ and $u \cdot \lambda = \mu \neq 0$. Then the cell in μ/λ corresponding to $a-1$ occurs directly above the cell in μ/λ corresponding to a .*

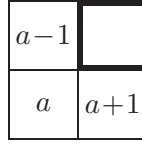
Proof. Recall that by (2.4) a generator u_i of \mathcal{A}_k acts on $(k+1)$ -cores by adding all available boxes of residue i . On the k -bounded partition this amounts to adding one box (which corresponds to the topmost added box on the core).

To show the claim of the lemma, we first show that in the core, the topmost added $a-1$ cannot be more than one square above the topmost added a . Suppose it were. In that case, we consider $\text{core}_{k+1}(\lambda)$ near the place where the topmost $a-1$ will be added. At that location, we must have the following configuration:



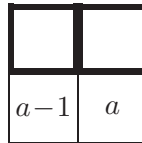
where the bold border represents a cell *not* present in $\text{core}_{k+1}(\lambda)$.

Furthermore, we must have the following configuration at the point where the topmost a will be added to $\text{core}_{k+1}(\lambda)$:



(The a to be added cannot be in the first column because it is below the diagram above). But this means there is a removable border strip from the a in the first diagram to the $a-1$ in the second diagram. The length of this strip is a multiple of $k+1$, which is a contradiction with being a $(k+1)$ -core. So the topmost added $a-1$ cannot be more than one cell above the topmost added a .

Now we show that the topmost added $a-1$ cannot be below the topmost added a . Again, assume the contrary. After adding the cells corresponding to wa , the core must be in the following configuration near the topmost added a :



(In particular, the cell above the a cannot be an addable cell, since we are assuming the topmost added $a-1$ will be below the topmost added a .) At the same time, the border

near where the topmost $a - 1$ will be added must look like this:

$a - 2$	
$a - 1$	a

Thus we have a removable border strip from the $a - 1$ in the top diagram to the $a - 2$ in the bottom diagram, whose length is again a multiple of $k + 1$.

We now have that, when multiplying a core by $(a - 1)wa$, the topmost added $a - 1$ must sit directly above the topmost added a . Therefore the cells added to the partition are necessarily in consecutive rows. It remains to verify that they are in the same column. For this, we appeal to the bijection core_{k+1} between $(k + 1)$ -cores and k -bounded partitions. Let y, y' be the rows of the topmost added $a - 1$ and a , respectively. The boxes added to the partition will end up in different columns if and only if the number of cells in y with hook length greater than $k + 1$ is different from the number of cells in y' with hook length greater than $k + 1$. But every pair of adjacent cells with one from y and one from y' have hook-lengths differing by exactly 1. Since no cell in a $(k + 1)$ -core can have hook length exactly $k + 1$, every vertical domino in the rows y, y' is either completely destroyed or completely preserved under the bijection, proving the assertion. \square

Ultimately, we are interested in k -connected hook words w . Such words can be written as $w = HV$, where H is a horizontal strip (strictly decreasing), V is a vertical strip (strictly increasing), and the smallest letter in w is part of V . The strict increase/decrease follows from the fact that we are working in the affine nilCoxeter algebra and hence consecutive repeated letters annihilate any partition. We can further factor V into maximal segments $v^{(i)}$ consisting of consecutive letters as

$$(4.1) \quad V = v^{(j)}v^{(j-1)} \dots v^{(1)}.$$

Corollary 4.3. *Let $\lambda \in \mathcal{P}^{(k)}$ and $V = v^{(j)}v^{(j-1)} \dots v^{(1)}$ as described above, such that $V \cdot \lambda \neq 0$. Then each $v^{(i)}$ adds a connected vertical strip to $(v^{(i-1)}v^{(i-2)} \dots v^{(1)}) \cdot \lambda$. Furthermore, these strips are disjoint from one another; that is, $(V \cdot \lambda)/\lambda$ consists of j connected components.*

Proof. Let $v^{(i)} = (a - s) \dots (a - 1)a$. Each pair $(a - r - 1)(a - r)$ for $0 \leq r < s$ must correspond to a vertical domino in the skew k -bounded partition by Lemma 4.2. This proves the first statement.

Now, consider any two sections of this word $v^{(\ell)}, v^{(i)}$ with $\ell > i$. First note that no core-cell added by $v^{(\ell)}$ can be in any of the rows containing the topmost vertical strip corresponding to $v^{(i)}$: this is because the lowest addable residue in these rows is $a + 1$ which cannot appear in $v^{(\ell)}$ by the verticality of V . Thus we only have to consider the case where the bottom of the topmost vertical strip added by $v^{(\ell)}$ occurs in the row immediately above the topmost $a - s$ added by $v^{(i)}$. But notice that the cells of residue $a - s$ added by $v^{(i)}$ to the core have remained as a removable residue: no residue $a - s + 1$ or $a - s - 1$ has

been added since. Thus the corresponding partition cell must remain as a removable cell in the partition. \square

Given a skew partition $(V \cdot \lambda)/\lambda$ of the form of the previous corollary, we call the cell directly above the vertical strip corresponding to $v^{(i)}$ a *cap* for $v^{(i)}$. In the following lemma, we show that if we apply a k -connected hook word to λ , every vertical strip except the last will have a cap.

Lemma 4.4. *Let $w = HV$ be a k -connected hook word with V as in (4.1). Suppose $v^{(i)} = a(a+1) \cdots (a+s)$ for a fixed i . If $i = j$, then $v^{(j)}$ does not have a cap. If $i < j$, then*

- (1) $a - 1$ occurs as a letter in H , and
- (2) this $a - 1$ forms a cap for $v^{(i)}$.

Proof. First assume that $i < j$. Since V is a vertical strip and $v^{(i)}$ is of maximal length, the letter $(a - 1)$ cannot appear anywhere in V . Since $v^{(i+1)}$ consists of letters smaller than a , and w is k -connected, it must be the case that $(a - 1)$ appears somewhere in H . Note that the letters appearing between the a in $v^{(i)}$ and the $a - 1$ in H are all strictly less than $a - 1$. Hence by Lemma 4.2, $a - 1$ must be a cap for $v^{(i)}$.

We now show that $v^{(j)}$ does not have a cap. In this case a is the smallest letter in w . Therefore, Ha is a horizontal strip in the partition, and in particular, the cell directly above the cell corresponding to a does not appear. \square

We can now state precisely what happens to the height statistic when we apply a k -connected hook word to a partition λ .

Proposition 4.5. *Let $\mu, \lambda \in \mathcal{P}^{(k)}$, such that $\mu = w \cdot \lambda$ for a k -connected hook word w of length $r \leq k$. Then*

- (1) $\text{asc}(w) = \text{ht}(\mu/\lambda)$;
- (2) $\text{ht}(\mu/\lambda) + \text{ht}(\mu^{(k)}/\lambda^{(k)}) = r - 1$.

Proof. Factoring w as $w = HV$ with the smallest letter of w in V , we have $\text{asc}(w) = \text{len}(V) - 1$ by definition. By Corollary 4.3, we have $\text{ht}((V \cdot \lambda)/\lambda) = \text{len}(V) - j$. Because the cells added by H form a horizontal strip, a cell c created by H will only increase the height statistic if it forms the cap for some $v^{(i)}$. By Lemma 4.4, $j - 1$ of the vertical strips will obtain a cap when applying the word H . Hence

$$\text{ht}(\mu/\lambda) = \text{ht}((V \cdot \lambda)/\lambda) + (j - 1) = \text{len}(V) - j + j - 1 = \text{len}(V) - 1 = \text{asc}(w)$$

proving (1).

To prove (2), let w' be the image of w under the map which replaces every letter i by $k + 1 - i \pmod{k + 1}$. It is easy to see that $w' \cdot \lambda^{(k)} = \mu^{(k)}$ if $w \cdot \lambda = \mu$. Also, if $w = HV$ as a hook word, then $w' = V'H'$ with $\text{len}(V') = \text{len}(H) + 1$ and $\text{len}(H') = \text{len}(V) - 1$, grouping the largest letter with V' . By Lemma 4.1 (1), the word w' is equivalent to a hook word $w'' = H''V''$ with $\text{len}(H'') = \text{len}(H')$ and $\text{len}(V'') = \text{len}(V')$. Hence, applying part (1) to $\lambda^{(k)}, \mu^{(k)}, w''$ we conclude that

$$\text{asc}(w'') = \text{ht}(\mu^{(k)}/\lambda^{(k)}).$$

This implies

$$\begin{aligned} \text{ht}(\mu/\lambda) + \text{ht}\left(\mu^{(k)}/\lambda^{(k)}\right) &= \text{asc}(w) + \text{asc}(w'') = \text{len}(V) - 1 + \text{len}(V'') - 1 \\ &= \text{len}(V) - 1 + \text{len}(H) = r - 1. \quad \square \end{aligned}$$

We are finally in the position to show that the conditions of Theorem 3.1 imply the conditions of Definition 1.1.

Proposition 4.6. *If the pair (w, μ) satisfies (1') through (4') of Theorem 3.1, then μ must satisfy (0) through (4) of Theorem 1.2.*

Proof. By the k -Pieri rule, each letter in w adds one cell to λ and $\lambda^{(k)}$, which ensures the containment condition (0). Since each letter in w adds one box to the k -bounded partition λ , condition (1) immediately follows from (1'). Condition (3) is a direct translation of condition (3') on the level of cores.

To see (2), note that for $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ to contain a 2×2 square, the word w such that $\mu = w \cdot \lambda$ must contain the pattern $a(a+1)(a-1)a$ or $a(a-1)(a+1)a$ (meaning that these have to appear as subwords of w). However, these patterns cannot appear in hook words, a contradiction. Hence (2) follows from (2').

Condition (4) follows immediately from Proposition 4.5 (2). \square

4.2. Unprimed implies primed. We now show that the conditions of Definition 1.1 imply those of Theorem 3.1. We first show that with the conditions of Definition 1.1, there is indeed a word w for an element in the affine nilCoxeter algebra such that $\mu = w \cdot \lambda$.

Lemma 4.7. *Let $r \leq k$, and the pair λ, μ satisfies conditions (0) through (4) of Definition 1.1. Then there exists a k -connected hook word w of length r such that $w \cdot \lambda = \mu$.*

Proof. The proof proceeds as follows. We first produce a k -connected hook word w such that $w \cdot \lambda = \nu \supseteq \mu$. Then we show that

$$\text{ht}(\nu/\lambda) + \text{ht}\left(\nu^{(k)}/\lambda^{(k)}\right) \leq \text{ht}(\mu/\lambda) + \text{ht}\left(\mu^{(k)}/\lambda^{(k)}\right) = r - 1.$$

On the other hand, we can appeal to Proposition 4.5 (2) to conclude that

$$(4.2) \quad \text{ht}(\nu/\lambda) + \text{ht}\left(\nu^{(k)}/\lambda^{(k)}\right) = \text{len}(w) - 1.$$

Since $\nu \supseteq \mu$, we know that $\text{len}(w) \geq r$. Combining this with the equations above, we conclude that $\text{len}(w) = r$. Hence $|\nu| = |\mu| = |\lambda| + r$ and, since $\nu \supseteq \mu$, we must have $\nu = \mu$. Thus $w \cdot \lambda = \mu$.

We now give the details of the construction of the word w . Let \mathcal{I} be the set of all residues which appear in $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$, which is an interval due to k -connectedness. Let \mathcal{V} be the set of residues occurring in the cells of $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ which are not topmost in their column. Let $V = v_a \cdots v_0$ be the “vertical strip word” consisting of the elements of \mathcal{V} . That is, $v_a < v_{a-1} < \cdots < v_0$ with respect to the canonical order of \mathcal{I} , and $\mathcal{V} = \{v_i\}_{i=0}^a$.

We claim that $V \cdot \lambda \neq 0$ and furthermore that $\text{core}_{k+1}(V \cdot \lambda)$ contains all of the cells in $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ which are not topmost in their column. Note that any cell which is

not topmost in its column must be leftmost in its row, since the skew core is a ribbon. If i is the residue of any such cell, it will be an addable residue when the residues of all cells below it (in its column) have been added. But these are necessarily all larger than i in the canonical order.

Now let \mathcal{H} be the set of residues occurring in the cells of $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$, which do not occur in $V \cdot \lambda$. Let $H = h_b \cdots h_0$ be the “horizontal strip word” consisting of the elements of \mathcal{H} . That is, $h_b > h_{b-1} > \cdots > h_0$ with respect to the canonical order of \mathcal{I} , and $\mathcal{H} = \{h_i\}_{i=0}^b$.

We define $w = HV$ and claim that $\nu := w \cdot \lambda \supseteq \mu$. By construction of V , any cell in $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ which is not part of $\text{core}_{k+1}(V \cdot \lambda)$ must be topmost in its column. Therefore, if i is the residue of any such cell, it will be an addable residue when the residues of all cells to the left of it (in its row) have been added. But these are necessarily all smaller than i in the canonical order.

Notice that the residues appearing in w are precisely those occurring in \mathcal{I} , so w is k -connected. By construction, w is also a hook word (although note that the smallest letter of w is part of H , not V , according to this construction). Since $\nu \supseteq \mu$, we must have $\text{len}(w) \geq r$. By Proposition 4.5 (2), we conclude that (4.2) holds.

We now claim that

$$\text{ht}(\nu/\lambda) \leq \text{ht}(\mu/\lambda).$$

The proof proceeds as follows. First we note that by Proposition 4.5 (1), we have that $\text{ht}(\nu/\lambda) = \text{asc}(w) = \text{len}(V)$. Next we observe that from the definition of the ht statistic that $\text{ht}(\mu/\lambda)$ is the number of vertical dominos in μ/λ since μ/λ is a ribbon. Next we recall that by definition, every letter a in V corresponds to a vertical domino in $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ (where a is the residue of the bottom of the domino). It remains to show that each of these dominos corresponds to a domino in μ/λ . We first show that if $(a, a-1)$ occur as residues of a vertical domino in the skew core, then the topmost occurrence of a and the topmost occurrence of $a-1$ occur as a vertical domino in the skew core. Suppose there were an $a-1$ occurring above the topmost domino. Then there would be an a below it in $\text{core}_{k+1}(\lambda)$, and there would also be an $a-1$ in $\text{core}_{k+1}(\lambda)$ to the left of the a in the domino. This would form a forbidden border strip in $\text{core}_{k+1}(\lambda)$. Now suppose instead there were an a occurring above the topmost domino. This a to the $a-1$ in the domino would form a forbidden border strip in $\text{core}_{k+1}(\mu)$.

Finally, we need to show that the topmost domino in $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ with residue $(a, a-1)$ corresponds to a domino in μ/λ . Suppose that these are in row i and $i+1$, respectively. Recall that to go from a $(k+1)$ -core to a k -bounded partition one crosses out all cells with hook length greater than $k+1$. Since $\text{core}_{k+1}(\lambda)_i = \text{core}_{k+1}(\lambda)_{i+1}$ and there are no cells with hook length $k+1$ in a $(k+1)$ -core, the rightmost crossed out cell in $\text{core}_{k+1}(\lambda)$ under core_{k+1}^{-1} in rows i and $i+1$ must be in the same column j and hence we must have $\lambda_i = \lambda_{i+1}$. This implies that the hook length of the cell $(i+1, j+1)$ in $\text{core}_{k+1}(\lambda)$ is strictly smaller than k . Now look at the hook length of the cell $(i+1, j+1)$ in $\text{core}_{k+1}(\mu)$. Since $\text{core}_{k+1}(\mu)/\text{core}_{k+1}(\lambda)$ is a ribbon, there is precisely one cell in row $i+1$ in this skew shape. Any cells in column $j+1$ that are in $\text{core}_{k+1}(\mu)$, but not in $\text{core}_{k+1}(\lambda)$,

must have residues different from a and $a - 1$ since the ones in row i and $i + 1$ are topmost. This implies that the hook length of $(i + 1, j + 1)$ in $\text{core}_{k+1}(\mu)$ is still smaller than $k + 1$, so that there is a cell in row $i + 1$ of μ/λ . Since μ is a partition and $\lambda_i = \lambda_{i+1}$, there is also a cell in row i of μ/λ , so that there is a domino as desired.

The same argument applied to the conjugate partitions shows

$$\text{ht} \left(\nu^{(k)} / \lambda^{(k)} \right) \leq \text{ht} \left(\mu^{(k)} / \lambda^{(k)} \right).$$

From this we get that $\text{len}(w) \leq r$, and we can complete the proof as outlined in the first paragraph. \square

Proposition 4.8. *Fix $\lambda \in \mathcal{P}^{(k)}$. If $\mu \in \mathcal{P}^{(k)}$ satisfies conditions (0) through (4) of Definition 1.1, then there exists a unique word w so that the pair (w, μ) satisfies the conditions (1') through (4') of Theorem 3.1.*

Proof. By Lemma 4.7, there exists a k -connected hook word w of length r such that $\mu = w \cdot \lambda$. This implies the existence of the pair (μ, w) satisfying (1') through (4') of Theorem 3.1. By Lemma 4.1 (2) this pair is unique. \square

By Propositions 4.6 and 4.8, the summations in Theorems 1.2 and 3.1 are the same. By Proposition 4.5 (1) the signs also agree. Hence Theorem 3.1 implies Theorem 1.2.

5. OUTLOOK

By Corollaries 1.3, 1.4 and 1.5, the Murnaghan-Nakayama rule proved in this paper gives the expansion of the power sum symmetric functions in terms of the k -Schur functions $s_\lambda^{(k)} \in \Lambda_{(k)}$ and the expansion of the dual k -Schur functions $\mathfrak{S}_\lambda^{(k)} \in \Lambda^{(k)}$ in terms of the power sums:

$$p_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \chi_{\lambda, \nu}^{(k)} s_\lambda^{(k)} \quad \text{and} \quad \mathfrak{S}_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \chi_{\nu, \lambda}^{(k)} p_\lambda.$$

Unlike in the symmetric function case, where the Schur functions $s_\lambda \in \Lambda$ are self-dual, there should be a dual version of the Murnaghan-Nakayama rule of this paper, namely a combinatorial formula for the coefficients $\tilde{\chi}_{\lambda, \nu}^{(k)}$ in the expansion of the power sum symmetric functions in terms of the dual k -Schur functions

$$p_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \tilde{\chi}_{\lambda, \nu}^{(k)} \mathfrak{S}_\lambda^{(k)}$$

or, equivalently by the same arguments as in the proof of Corollary 1.5,

$$s_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \tilde{\chi}_{\nu, \lambda}^{(k)} p_\lambda.$$

Since the $s_\nu^{(k)}$ are known to be Schur-positive symmetric functions [7], they correspond to representations of the symmetric group under the Frobenius characteristic map. Furthermore, the characters of these representations are given by the $\tilde{\chi}_{\nu, \lambda}^{(k)}$. An explicit description

of such representations is an interesting open problem, which has been studied by Li-Chung Chen and Mark Haiman [1]. In the most generality they conjecture a representation theoretical model for the k -Schur functions with a parameter t which keeps track of the degree grading; the $\tilde{\chi}_{\nu,\lambda}^{(k)}$ described above should give the characters of these representations without regard to degree. Tables of $\chi_{\lambda,\mu}^{(k)}$ and $\tilde{\chi}_{\lambda,\mu}^{(k)}$ are listed in Appendices A and B.

Computer evidence suggests that the ribbon condition (2) of Definition 1.1 might be superfluous because it is implied by the other conditions of the definition. This was checked for $k, r \leq 11$ and for all $|\lambda| = n \leq 12$ and $|\mu| = n + r$.

APPENDIX A. TABLES OF $\chi_{\lambda,\nu}^{(k)}$

In the tables below, the partitions λ index the row and ν indexes the column for the values of $\chi_{\lambda,\nu}^{(k)}$.

$$k = 2, n = 3$$

	(111)	(21)
(111)	1	-1
(21)	1	1

$$k = 2, n = 4$$

	(1111)	(211)	(22)
(1111)	1	-1	1
(211)	2	0	-2
(22)	1	1	1

$$k = 2, n = 5$$

	(11111)	(2111)	(221)
(11111)	1	-1	1
(2111)	2	0	-2
(221)	1	1	1

$$k = 2, n = 6$$

	(111111)	(21111)	(2211)	(222)
(111111)	1	-1	1	-1
(21111)	3	-1	-1	3
(2211)	3	1	-1	-3
(222)	1	1	1	1

$$k = 3, n = 4$$

	(1111)	(211)	(22)	(31)
(1111)	1	-1	1	1
(211)	2	0	-2	-1
(22)	2	0	2	-1
(31)	1	1	1	1

$k = 3, n = 5$

	(11111)	(2111)	(221)	(311)	(32)
(11111)	1	-1	1	1	-1
(2111)	3	-1	-1	0	2
(221)	4	0	0	-2	0
(311)	3	1	-1	0	-2
(32)	1	1	1	1	1

$k = 3, n = 6$

	(111111)	(21111)	(2211)	(3111)	(222)	(321)	(33)
(111111)	1	-1	1	1	-1	-1	1
(21111)	4	-2	0	1	2	1	-2
(2211)	4	0	0	-2	-4	0	1
(3111)	6	0	-2	0	0	0	3
(222)	4	0	0	-2	4	0	1
(321)	4	2	0	1	-2	-1	-2
(33)	1	1	1	1	1	1	1

$k = 4, n = 5$

	(11111)	(2111)	(221)	(311)	(32)	(41)
(11111)	1	-1	1	1	-1	-1
(2111)	3	-1	-1	0	2	1
(221)	5	-1	1	-1	-1	1
(311)	3	1	-1	0	-2	-1
(32)	5	1	1	-1	1	-1
(41)	1	1	1	1	1	1

$k = 4, n = 6$

	(111111)	(21111)	(2211)	(3111)	(222)	(321)	(411)	(33)	(42)
(111111)	1	-1	1	1	-1	-1	-1	1	1
(21111)	4	-2	0	1	2	1	0	-2	-2
(2211)	8	-2	0	-1	-2	1	2	-1	0
(3111)	6	0	-2	0	0	0	0	3	2
(222)	5	-1	1	-1	3	-1	1	2	-1
(321)	8	2	0	-1	2	-1	-2	-1	0
(411)	4	2	0	1	-2	-1	0	-2	-2
(33)	5	1	1	-1	-3	1	-1	2	-1
(42)	1	1	1	1	1	1	1	1	1

APPENDIX B. TABLES OF $\tilde{\chi}_{\lambda,\nu}^{(k)}$

In the tables below, the partitions λ index the row and ν indexes the column for the values of $\tilde{\chi}_{\lambda,\nu}^{(k)}$.

$$k = 2, n = 3$$

	(111)	(21)
(111)	3	-1
(21)	3	1

$$k = 2, n = 4$$

	(1111)	(211)	(22)
(1111)	6	-2	2
(211)	6	0	-2
(22)	6	2	2

$$k = 2, n = 5$$

	(11111)	(2111)	(221)
(11111)	30	-6	2
(2111)	30	0	-2
(221)	30	6	2

$$k = 2, n = 6$$

	(111111)	(21111)	(2211)	(222)
(111111)	90	-18	6	-6
(21111)	90	-6	-2	6
(2211)	90	6	-2	-6
(222)	90	18	6	6

$$k = 3, n = 4$$

	(1111)	(211)	(22)	(31)
(1111)	4	-2	0	1
(211)	6	0	-2	0
(22)	2	0	2	-1
(31)	4	2	0	1

$$k = 3, n = 5$$

	(11111)	(2111)	(221)	(311)	(32)
(11111)	10	-4	2	1	-1
(2111)	10	-2	-2	1	1
(221)	10	0	2	-2	0
(311)	10	2	-2	1	-1
(32)	10	4	2	1	1

$k = 3, n = 6$

	(111111)	(21111)	(2211)	(3111)	(222)	(321)	(33)
(111111)	20	-8	4	2	0	-2	2
(21111)	40	-8	0	1	0	1	-2
(2211)	30	-2	2	-3	-6	1	0
(3111)	20	0	-4	2	0	0	2
(222)	30	2	2	-3	6	-1	0
(321)	40	8	0	1	0	-1	-2
(33)	20	8	4	2	0	2	2

$k = 4, n = 5$

	(11111)	(2111)	(221)	(311)	(32)	(41)
(11111)	5	-3	1	2	0	-1
(2111)	10	-2	-2	1	1	0
(221)	5	-1	1	-1	-1	1
(311)	10	2	-2	1	-1	0
(32)	5	1	1	-1	1	-1
(41)	5	3	1	2	0	1

$k = 4, n = 6$

	(111111)	(21111)	(2211)	(3111)	(222)	(321)	(411)	(33)	(42)
(111111)	15	-7	3	3	-3	-1	-1	0	1
(21111)	15	-5	-1	3	3	1	-1	0	-1
(2211)	25	-3	1	-2	-3	0	1	-2	1
(3111)	20	0	-4	2	0	0	0	2	0
(222)	5	-1	1	-1	3	-1	1	2	-1
(321)	25	3	1	-2	3	0	-1	-2	1
(411)	15	5	-1	3	-3	-1	1	0	-1
(33)	5	1	1	-1	-3	1	-1	2	-1
(42)	15	7	3	3	3	1	1	0	1

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